Poisson-based expectile regression for non-negative data with a mass-point at zero

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OLS and quantiles

- Consider a linear model of the form

$$y_i = x_i'\beta + \varepsilon_i,$$

  where the error ε_i is not independent of x_i.

- The standard way to learn about the effect of x_i on y_i is to assume that $\mathrm{E}\left[y_i|x_i\right]$ is linear and estimate the parameters by **least squares**

$$\hat{\beta} = \arg\min_b \frac{1}{n} \sum \left(y_i - x_i'b\right)^2 .$$

- We can also estimate **conditional quantiles**, $\mathrm{Q}_{y_i}\left[\alpha|x_i\right]$, using the method introduced by Koenker and Bassett (1978)

$$\tilde{\beta}\left(\alpha\right) = \arg\min_b \frac{1}{n} \left\{\sum\nolimits_{y_i \geq x_i'b} \alpha \left|y_i - x_i'b\right| + \sum\nolimits_{y_i < x_i'b} \left(1-\alpha\right) \left|y_i - x_i'b\right|\right\} .$$

- **Median regression** is a special case.
- Quantiles are **local** measures of location that depend on only on the properties of the distribution around the relevant quantile.

Expectiles: Introduction

- Newey and Powell (1987) introduced **expectile regressions**, whose parameters can be estimated by solving

$$\hat{\beta}(\tau) = \arg\min_{b} \frac{1}{n} \left\{ \sum_{y_i \geq x_i' b} \tau \left(y_i - x_i' b\right)^2 + \sum_{y_i < x_i' b} (1-\tau) \left(y_i - x_i' b\right)^2 \right\}.$$

- **Mean regression** (OLS) is a special case when $\tau = 0.5$.
- For any $\tau \in (0, 1)$, the expectile τ of x, denoted $\mathrm{E}_x(\tau)$, **can be interpreted** as the expectation of x in a population where values of x above the expectile occur $\tau / (1-\tau)$ times as often as they do in the population of interest.
- An **analogous** results holds for quantiles.
- Unlike most estimators, here the **estimator defines** the object being estimated.

Expectiles: Properties

- In the **unconditional** case, each expectile corresponds to a quantile, and vice-versa.
- However, except in special cases, there is **no correspondence** between conditional expectiles and **conditional** quantiles.
- **Like** quantiles, expectiles provide information on the **location of different regions** of the distribution of a variable.
- In **contrast** to quantiles, expectiles are **global** measures of location that depend on global properties of the distribution.
- Admittedly, the **interpretation** of expectiles is not as intuitive as that of quantiles.
- In general, expectiles have **no advantage** over quantiles and Roger Koenker's (2013) view is that "*Expectiles belong in the spittoon*."

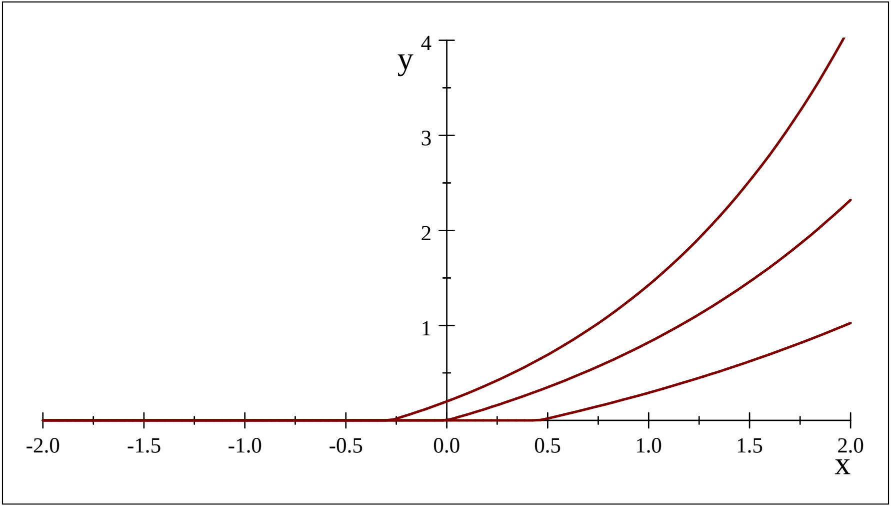
Quantiles vs. expectiles with non-negative data

- In many applications, the variable of interest takes only **non-negative** values and there is a mass-point at zero.
- We will look at the labour supply of married women (average hours per week) from the 1987 wave of PSID; this sample was used by Lee (1995).
- The table below displays some quantiles and expectiles for these data.

θ	0.01	0.05	0.10	0.25	0.50	0.75	0.90	0.95	0.99
$Q_x(\theta)$	0	0	0	0	25.1	37.4	40.1	40.1	53.9
$E_x(\theta)$	0.8	3.6	6.6	13.3	21.8	29.6	35.3	35.3	45.8

- Expectiles **smooth out** the mass point at zero.
- Because they are global measures of location, expectiles are **always positive**.

The trouble with quantiles with zeros



Setup and notation

- We consider a standard exponential model, typically used for this kind of data

$$y_i = \exp\left(x_i'\beta\right)\eta_i,$$

  where

  - y_i denotes the outcome of interest,
  - x_i is a vector of explanatory variables,
  - β is a conformable vector of parameters,
  - and η_i is a non-negative error term such that $\mathrm{E}\left(\eta_i|x_i\right) = 1$.

- Therefore $\mathrm{E}\left[y_i|x_i\right] = \exp\left(x_i'\beta\right)$.
- $\mathrm{E}\left(\eta_i|x_i\right) = 1$ but other features of its distribution may depend on x_i.
- In particular, η_i is generally **heteroskedastic**.

Expectiles

- As in Bergstrand, Clance and Santos Silva (2025), we **assume** that the τ-th conditional expectile of η_i has the form

$$\mathrm{E}_{\eta}\left(\tau|y_i\right) = \exp\left(x_i'\delta\left(\tau\right)\right).$$

  - The **exponential** function is used because all expectiles of η_i are positive.
  - The parameters are indexed by τ because they **vary** across expectiles.

- This setup implies that the **conditional expectiles** of y_i have the form

$$\mathrm{E}_{y}\left(\tau|x_i\right) = \exp\left(x_i'\beta\left(\tau\right)\right),$$

  with $\beta\left(\tau\right) = \beta + \delta\left(\tau\right)$.

  - x_i affects both the **mean** and the **dispersion** of y.
  - If η_i is **independent** of x_i, only the intercept changes with τ and all expectiles are proportional to each other.

The APPML estimator

- To estimate exponential expectiles we can use Efron's (1992) **asymmetric Poisson maximum likelihood estimator (APPML).**
- The estimator was intended for **count data** but can be used for other kinds of non-negative data.
- The APPML estimator of $\beta(\tau)$ based on a sample $\{(y_i, x_i)\}$ is the solution to moment conditions of the form:

$$\sum_{i=1}^{n} \omega_i \left(y_i - \exp\left(x_i'\hat{\beta}(\tau)\right)\right) x_i = 0,$$

  with

$$\omega_i = \left|\tau - \mathbf{1}\left(y_i < \exp\left(x_i'\hat{\beta}(\tau)\right)\right)\right|.$$

- This a Poisson regression that gives different **weights** to observations above or below the estimated expectile.
- The `appmlhdfe` command (Clance and Santos Silva, 2025) implements this estimator.

appmlhdfe

- `appmlhdfe` is based on the powerful `ppmlhdfe` command by Correia et al. (2019) and shares many of its functionalities.

Syntax

```
appmlhdfe depvar [indepvars] [if] [in] [, options]
```

`expectile(#)`: estimates # expectile; default is expectile(.5), which corresponds to Poisson regression.

`absorb`(varlist): categorical variables to be absorbed (fixed effects).

`residual`(varname): saves the residuals as varname.

`start`(varname): vector of residuals to be used as starting values.

Illustration

- Data on **labour supply** of married women (average hours per week) from the 1987 wave of PSID as used by Lee (1995).
- The independent variables are:
  - **education** in years (educ),
  - **age**, in years
  - number of **children** by age group (pkid, skid, hkid),
  - **race** (0 if white, 1 otherwise),
  - **home** (1 if owner, 0 otherwise),
  - **mort** (1 if mortgage on home, 0 otherwise),
  - husband's **occupation** dummies (manager, clerical, farmer),
  - local **unemployment** rate in percentage points (ur).
- We will ignore the **upper bound** on the number of hours per week, and estimate exponential models.

Results I

```
. appmlhdfe hours edu, a(age pkid skid hkid black ownh mort manager ///
> clerical farmer ur)

 Number of obs = 3373
 Iterations = 1
 Tolerance = 1.000e-07
 Objective function = 0
 % of negative residuals = .482
 R-squared: .19880988
.5 expectile regression
```

hours	Coef.	Robust Std. Err.	z	P>|z|	[95% Conf.	Interval]
edu	.0476741	.0064172	7.43	0.000	.0350965	.0602516
_cons	2.532422	.085541	29.60	0.000	2.364765	2.700079

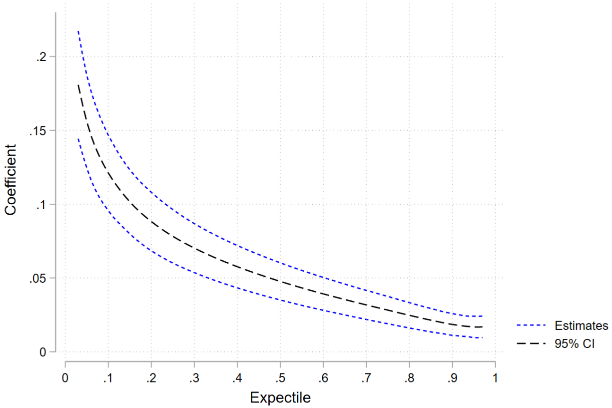
Results II

```
. appmlhdfe hours edu, a(age pkid skid hkid black ownh mort manager ///
> clerical farmer ur) e(.10)

Iteration 1: objective function = 8847.2249
Iteration 2: objective function = 29.708204
Iteration 3: objective function = .75390222
Iteration 4: objective function = .0000173
Iteration 5: objective function = 0

 Number of obs = 3373
 Iterations = 5
 Tolerance = 1.000e-07
 Objective function = 0
 % of negative residuals = .331
 R-squared: .17045263
.1 expectile regression
```

hours	Coef.	Robust Std. Err.	z	P>|z|	[95% Conf.	Interval]
edu	.1211607	.0129979	9.32	0.000	.0956854	.146636
_cons	.6872244	.1757405	3.91	0.000	.3427794	1.031669

Results III

Expectile	10th	25th	50th	75th	90th
Educ	0.121 (0.013)	0.078 (0.009)	0.048 (0.006)	0.028 (0.005)	0.019 (0.004)



- Educ **increases** the mean and **reduces** the dispersion of labour supply

Summary

- In most situations, expectiles are not particularly interesting.
- There are, however, cases where expectiles can be very useful.
- Here we considered that case of non-negative data with a mass-point at zero.
- Quantile regressions are not very appealing in this context.
- Expectiles provide an alternative way to study how the regressors affect different regions of the conditional distribution.
- The estimator is very easy to implement and the parameters have a straightforward interpretation.

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